Matrix elements of the plaquette operator of Lattice Gauge Theory

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Abstract

We show that in the *spin-network* basis it is possible to compute the matrix elements of any given operator of the Hamiltonian formulation of Lattice Gauge Theory (LGT). We give the explicit calculation for the case of the plaquette operator.

1 Introduction

In recent papers[1, 2] we proposed a group theoretical description of the Hilbert space for lattice gauge theories (LGT) in the Hamiltonian framework[3]. This approach, based on representation theory, allows to overcome the problem of selecting the gauge invariant Hilbert space. In particular, the difficulty of explicitly solving Mandelstam's identities[5] in the context of Wilson loops[4] is circumvented. Moreover, such an approach yields a general setting for the computation of the matrix elements of the relevant operators.

We will briefly sketch the main concepts underlying our construction and present a physically interesting application as an example.

2 The spin network basis

In $(d+1)$ dimensions the configuration space of LGT is defined associating gauge field variables $U_k(\mathbf{x}) \in G$ to each link $(\mathbf{x}, \mathbf{x} + a\mathbf{e}_k)$ of a hypercubic periodic lattice of period aL, with L a positive integer.

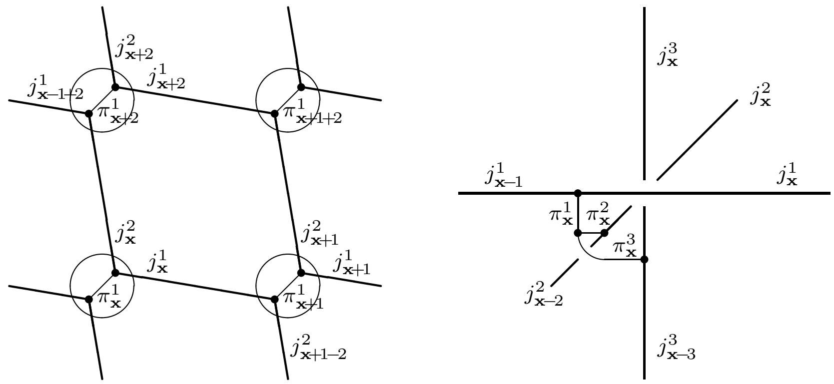

Figure 1: Graphical representation of (a possible) decomposition of a vertex in a two dimensional (left) and in a three dimensional (right) lattice. The solid dots denote the presence of a Wigner 3J symbol.

The corresponding quantum Hilbert space $\mathcal{H}$ is given by the gauge invariant square integrable functions $\psi(U) = \psi(U^\gamma) = \psi(\{U^\gamma_k(\mathbf{x})\})$ on the tensor product of $d \cdot L^d$ copies of the gauge group G. Gauge transformations act as $U_k(\mathbf{x}) \longrightarrow U^\gamma_k(\mathbf{x}) = \gamma^{-1}(\mathbf{x} + a\mathbf{e}_k)U_k(\mathbf{x})\gamma(\mathbf{x})$. The variables conjugated to $U_k(\mathbf{x})$ are the outgoing/ingoing electric fields $E^\alpha_{\pm k}(\mathbf{x})$ at the lattice point $\mathbf{x}$ in the directions $\mathbf{e}_{\pm k}$.

A classical result of representation theory[6] gives a way of constructing a basis of such Hilbert space. In fact, the set $\mathcal{RG} = \{\mathcal{R}^j \,|j \in J[G]\}$ of all the unitary inequivalent representations of a compact group G is numerable and all the representations $\mathcal{R}^j$ are finite dimensional. They are defined on the Hilbert space $\mathcal{H}^j$. Choosing an orthonormal basis for each representation $\mathcal{R}^j$, the matrix elements of a unitary operator $T(U)$ become $D^{(j)\alpha}{}_{\beta}(U)$ $(\alpha, \beta = 1, \dots, \dim(\mathcal{H}^j))$ of all the representations $\mathcal{R}^j$ are a numerable orthonormal basis of $\mathcal{L}^2[G, dU]$. This result, known as the Peter-Weyl theorem, implies that each vector of $\mathcal{H}$ can be written as

$$\psi(U) = \prod_{\mathbf{x}} \prod_{k=1}^{d} \sum_{j^k_{\mathbf{x}} \in J[G]} \sum_{\alpha^k_{\mathbf{x}}, \beta^k_{\mathbf{x}} = 1}^{\dim(j^k_{\mathbf{x}})} \left[\, D^{(j^k_{\mathbf{x}})\alpha^k_{\mathbf{x}}}{}_{\beta^k_{\mathbf{x}}}(U) \times c^{(j_{(1)}\cdots j_{(N_{lk})})\beta_{(1)}\cdots\beta_{(N_{lk})}}{}_{\alpha_{(1)}\cdots\alpha_{(N_{lk})}} \right], \tag{1}$$

where only gauge invariant combinations should be taken into account.

The implementation of gauge invariance turns into a set of constraints on the coefficients c. The c's factorize as products of group invariant tensors (intertwining operators) associated to the different lattice sites $\mathbf{x}$. By definition, an operator I connecting the Hilbert space of two representations, $\mathcal{R}$ and $\mathcal{R}'$, is an intertwining operator if $\mathsf{I} \cdot T(U) = T'(U) \cdot \mathsf{I}$, for every U in G. The set of all intertwining operators $\mathcal{I}(\mathcal{R}, \mathcal{R}')$ is a vector subspace of all the linear operators connecting the Hilbert space of the two representations $\mathcal{R}$ and $\mathcal{R}'$. This gives the coordinate free definition of the generalized Clebsh-Gordan coefficients of Yutsis-Levinson-Vanagas which are the matrix elements of these operators on the chosen basis. The integral of the product of K representations decomposes according to

$$\int dU \prod_{k=1}^{K} D^{(j_k)\alpha_k}{}_{\beta_k}(U) = \sum_{\pi} \frac{\mathsf{I}^{[\pi](j_1\dots j_K)}{}_{\beta_1\dots\beta_K} \, \mathsf{I}_{[\pi](j_1\dots j_K)}{}^{\alpha_1\dots\alpha_K}}{\mathsf{I}^{[\pi](j_1\dots j_K)}{}_{\gamma_1\dots\gamma_K} \, \mathsf{I}_{[\pi](j_1\dots j_K)}{}^{\gamma_1\dots\gamma_K}}, \tag{2}$$

where $\mathsf{I}_{[\pi](j_1\dots j_K)} \in \mathcal{I}(\mathcal{R}^{j_1} \otimes \dots \otimes \mathcal{R}^{j_K}, \emptyset)$, $\mathsf{I}^{[\pi](j_1\dots j_K)}$ is its adjoint and the sum is extended over a complete orthogonal basis of $\mathcal{I}(\mathcal{R}^{j_1} \otimes \dots \otimes \mathcal{R}^{j_K}, \emptyset)$.

Summarizing, Peter-Weyl theorem together with gauge invariance lead to the *spin network* basis elements

$$\psi_{\vec{\jmath},\vec{\pi}}(U) = \prod_{\mathbf{x}} \prod_{k=1}^{d} \sum_{\alpha^k_{\mathbf{x}},\beta^k_{\mathbf{x}}=1}^{\dim(j^k_{\mathbf{x}})} \left[D^{(j^k_{\mathbf{x}})\alpha^k_{\mathbf{x}}}{}_{\beta^k_{\mathbf{x}}}(U_k(\mathbf{x})) \cdot \mathsf{I}_{\mathbf{x}}^{[\pi_{\mathbf{x}}]\,(j^1_{\mathbf{x}-1},\dots,j^d_{\mathbf{x}-d})\,\beta^1_{\mathbf{x}},\dots,\beta^d_{\mathbf{x}}}{}_{\alpha^1_{\mathbf{x}-1},\dots,\alpha^d_{\mathbf{x}-d}\,(j^1_{\mathbf{x}},\dots,j^d_{\mathbf{x}})} \right] . \tag{3}$$

The only non trivial part in this characterization is the choice of a convenient basis for the intertwining matrices. A natural choice is to use an orthonormal *spin network* basis. This is equivalent to the choice of an orthonormal basis for the intertwining operator

$$\sum_{\alpha^k_{\mathbf{x}},\beta^k_{\mathbf{x}}=1}^{\dim(j^k_{\mathbf{x}})} \mathsf{I}_{\mathbf{x}}^{[\pi_{\mathbf{x}}]\,(j^1_{\mathbf{x}-1},\dots,j^d_{\mathbf{x}-d})\,\beta^1_{\mathbf{x}},\dots,\beta^d_{\mathbf{x}}}{}_{\alpha^1_{\mathbf{x}-1},\dots,\alpha^d_{\mathbf{x}-d}\,(j^1_{\mathbf{x}},\dots,j^d_{\mathbf{x}})} \cdot \mathsf{I}^{\mathbf{x}}{}_{[\pi'_{\mathbf{x}}]}{}^{\alpha^1_{\mathbf{x}-1},\dots,\alpha^d_{\mathbf{x}-d}\,(j^1_{\mathbf{x}},\dots,j^d_{\mathbf{x}})}{}_{(j^1_{\mathbf{x}-1},\dots,j^d_{\mathbf{x}-d})\,\beta^1_{\mathbf{x}},\dots,\beta^d_{\mathbf{x}}} = \delta^{[\pi_{\mathbf{x}}]}_{[\pi'_{\mathbf{x}}]} . \tag{4}$$

As an example, for $SU(2)$ gauge group in $d = 2$ dimensions the spin-network basis elements can be characterized by three half integers associated to each lattice site. On the other hand, in $d = 3$ dimensions six half integers are needed per lattice site (see Fig. 1).

3 Matrix elements of the plaquette operator

Using the integral (2), it is straightforward to compute the action of the plaquette operator on the *spin-network basis*: just take the trace of the intertwining operators. More specifically, this consists of the evaluation of specific Wigner's nJ-symbols, which, in turn, involve traces of $6J$-symbols. The final result is:

$$\begin{aligned} \langle \vec{\jmath}\,', \vec{\pi}' | U_{\mathbf{y},r,s} | \vec{\jmath}, \vec{\pi} \rangle &= \int \prod_{\mathbf{x},k} dU_k(\mathbf{x}) \; \overline{\psi_{\vec{\jmath}\,',\vec{\pi}'}(U)} \; \psi_{\vec{\jmath},\vec{\pi}}(U) \;\cdot \\ &\quad \cdot U_s^{-1}{}^{\tau_1}{}_{\tau_4}(\mathbf{y}) U_r^{-1}{}^{\tau_4}{}_{\tau_3}(\mathbf{y}+a\mathbf{e}_s) U_s{}^{\tau_3}{}_{\tau_2}(\mathbf{y}+a\mathbf{e}_r) U_r{}^{\tau_2}{}_{\tau_1}(\mathbf{y}) = \\ &= \int \prod_{\mathbf{x}} \prod_{k=1}^{d} dU_k(\mathbf{x}) \sum_{\bar\alpha^k_{\mathbf{x}},\bar\beta^k_{\mathbf{x}}=1}^{\dim(j^k_{\mathbf{x}})} \sum_{\alpha^k_{\mathbf{x}},\beta^k_{\mathbf{x}}=1}^{\dim(j^{k\prime}_{\mathbf{x}})} D^{(j^k_{\mathbf{x}})\bar\beta^k_{\mathbf{x}}}{}_{\bar\alpha^k_{\mathbf{x}}}(U_k^{-1}(\mathbf{x})) \; D^{(j^{k\prime}_{\mathbf{x}})\alpha^k_{\mathbf{x}}}{}_{\beta^k_{\mathbf{x}}}(U_k(\mathbf{x})) \cdot \\ &\quad \cdot U_s^{-1}{}^{\tau_1}{}_{\tau_4}(\mathbf{y}) U_r^{-1}{}^{\tau_4}{}_{\tau_3}(\mathbf{y}+a\mathbf{e}_s) U_s{}^{\tau_3}{}_{\tau_2}(\mathbf{y}+a\mathbf{e}_r) U_r{}^{\tau_2}{}_{\tau_1}(\mathbf{y}) \cdot \\ &\quad \cdot \mathsf{I}_{\mathbf{x}}^{[\pi_{\mathbf{x}}]\,(j^1_{\mathbf{x}-1},\dots,j^d_{\mathbf{x}-d})\,\bar\beta^1_{\mathbf{x}},\dots,\bar\beta^d_{\mathbf{x}}}{}_{\bar\alpha^1_{\mathbf{x}-1},\dots,\bar\alpha^d_{\mathbf{x}-d}\,(j^1_{\mathbf{x}},\dots,j^d_{\mathbf{x}})} \cdot \mathsf{I}^{\mathbf{x}}{}_{[\pi'_{\mathbf{x}}]}{}^{\alpha^1_{\mathbf{x}-1},\dots,\alpha^d_{\mathbf{x}-d}\,(j^{1\prime}_{\mathbf{x}},\dots,j^{d\prime}_{\mathbf{x}})}{}_{(j^{1\prime}_{\mathbf{x}-1},\dots,j^{d\prime}_{\mathbf{x}-d})\,\beta^1_{\mathbf{x}},\dots,\beta^d_{\mathbf{x}}} \end{aligned} \tag{5}$$

Because of the traces, the choice of an explicit basis is irrelevant, as expected. All the indices of the intertwiner matrix elements are traced over their complex conjugate, except the contractor in the lattice points $\mathbf{y}$, $\mathbf{y}+r$, $\mathbf{y}+s$ and $\mathbf{y}+r+s$. The corresponding matrix elements are

$$\langle \vec{\jmath}\,', \vec{\pi}' | U_{\mathbf{y},r,s} | \vec{\jmath}, \vec{\pi} \rangle = \text{[Eq. (26) of ref. 1]} . \tag{6}$$

Notice that once the intertwining matrices are specified, i.e., when the Clebsh-Gordan coefficients are explicitly given, the matrix elements are known. In this way we have reduced the problem of the computation of the matrix elements of the plaquette operator to the computation of the trace of intertwining operators, i.e., of the trace of generalized Clebsh-Gordan coefficients. Now, it is well known that this is nothing more than the evaluation of specific Wigner's nJ-symbols and this can be always reduced to the computation of a Wigner's $6J$-symbol.

In the case of SU(2) the Wigner's $6J$-symbols are completely known and it is possible to find algebraic expressions for such matrix elements. In particular, in 2+1 dimensions and using the corresponding basis of Fig. 1 the matrix elements of the plaquette $U_{\mathbf{y},1,2}$ can be given. They are different from zero only if all the six primed and un-primed $j^1_{\mathbf{x}}$, $j^2_{\mathbf{x}}$, $\pi^1_{\mathbf{x}+1}$, $j^2_{\mathbf{x}+1}$, $\pi^1_{\mathbf{x}+2}$, $j^1_{\mathbf{x}+2}$, differ by a half integer for $\mathbf{x}=\mathbf{y}$. Their explicit expression is

$$\langle \vec{\jmath}\,', \vec{\pi}' | U_{\mathbf{y},1,2} | \vec{\jmath}, \vec{\pi} \rangle = \frac{(-1)^{\sum_{i=1}^{n}\left(|\epsilon_i - \epsilon_{i+1}| + \frac{C^i_{\mathbf{y}}}{2}\right)}}{\sqrt{\prod_{i=1}^{n}\left(2X^i_{\mathbf{y}}+1\right)\left(2Y^i_{\mathbf{y}}+1\right)}} \prod_{i=1}^{n} R\left[\begin{array}{cc} X^i_{\mathbf{y}} & X^{i+1}_{\mathbf{y}} \\ Y^i_{\mathbf{y}} & Y^{i+1}_{\mathbf{y}} \end{array}, C^i_{\mathbf{y}}\right]$$

where $\epsilon_i = X^i_{\mathbf{y}} - Y^i_{\mathbf{y}} = \pm\frac{1}{2}$,

$$\begin{array}{llllll} X^1_{\mathbf{y}} = j^1_{\mathbf{x}}\,, & X^2_{\mathbf{y}} = j^2_{\mathbf{x}}\,, & X^3_{\mathbf{y}} = \pi^1_{\mathbf{x}+2}\,, & X^4_{\mathbf{y}} = j^1_{\mathbf{x}+2}\,, & X^5_{\mathbf{y}} = j^2_{\mathbf{x}+1}\,, & X^6_{\mathbf{y}} = \pi^1_{\mathbf{x}+1} \\ Y^1_{\mathbf{y}} = j^{1\prime}_{\mathbf{x}}\,, & Y^2_{\mathbf{y}} = j^{2\prime}_{\mathbf{x}}\,, & Y^3_{\mathbf{y}} = \pi^{1\prime}_{\mathbf{x}+2}\,, & Y^4_{\mathbf{y}} = j^{1\prime}_{\mathbf{x}+2}\,, & Y^5_{\mathbf{y}} = j^{2\prime}_{\mathbf{x}+1}\,, & Y^6_{\mathbf{y}} = \pi^{1\prime}_{\mathbf{x}+1} \\ C^1_{\mathbf{y}} = \pi^1_{\mathbf{x}}\,, & C^2_{\mathbf{y}} = j^1_{\mathbf{x}-1+2}\,, & C^3_{\mathbf{y}} = j^2_{\mathbf{x}+2}\,, & C^4_{\mathbf{y}} = \pi^1_{\mathbf{x}+1+2}\,, & C^6_{\mathbf{y}} = j^1_{\mathbf{x}+1}\,, & C^6_{\mathbf{y}} = j^2_{\mathbf{x}+1-2}\,, \end{array}$$

and $R\left[\begin{array}{cc} X^i_{\mathbf{y}} & X^{i+1}_{\mathbf{y}} \\ Y^i_{\mathbf{y}} & Y^{i+1}_{\mathbf{y}} \end{array}, C^i_{\mathbf{y}}\right]$ is equal to

$$\begin{cases} \sqrt{\frac{1-2C^i_{\mathbf{y}}+X^i_{\mathbf{y}}+X^{i+1}_{\mathbf{y}}+Y^i_{\mathbf{y}}+Y^{i+1}_{\mathbf{y}}}{2}\;\frac{3+2C^i_{\mathbf{y}}+X^i_{\mathbf{y}}+X^{i+1}_{\mathbf{y}}+Y^i_{\mathbf{y}}+Y^{i+1}_{\mathbf{y}}}{2}} & \text{if } |\epsilon_i - \epsilon_{i+1}| = 0 \\ \sqrt{\frac{1+2C^i_{\mathbf{y}}+X^i_{\mathbf{y}}-X^{i+1}_{\mathbf{y}}+Y^i_{\mathbf{y}}-Y^{i+1}_{\mathbf{y}}}{2}\;\frac{1+2C^i_{\mathbf{y}}-X^i_{\mathbf{y}}+X^{i+1}_{\mathbf{y}}-Y^i_{\mathbf{y}}+Y^{i+1}_{\mathbf{y}}}{2}} & \text{if } |\epsilon_i - \epsilon_{i+1}| = 1 \end{cases}$$

A straightforward application of this result is the computation of the matrix elements of the LGT Hamiltonian operator

$$\hat{H} = \frac{g^2}{2a^{d-2}} \sum_{\mathbf{x},k} q_{\alpha\beta} E^\alpha_k(\mathbf{x}) E^\beta_k(\mathbf{x}) + \sum_P \frac{a^{d-4}}{g^2}\left[1 - \frac{U_P + U^*_P}{2\dim(U)}\right], \tag{7}$$

where $q_{\alpha\beta}$ is the Cartan metric. The sum over P runs over all unoriented plaquettes, U_P being the plaquette variable.

In fact, the basis vectors (3) are eigenstates of the kinetic term, while the potential (magnetic) term is realized as a multiplicative operator. Namely

$$\begin{aligned} \langle \vec{\jmath}\,', \vec{\pi}' | \hat{H} | \vec{\jmath}, \vec{\pi} \rangle = & \left(\frac{g^2}{2\; a^{d-2}} \sum_{\mathbf{x}} \sum_{k=1}^{d} C_2[j^2_{\mathbf{x}}] \; + \frac{a^{d-4}}{g^2} N_P \right) \delta^{\vec{\jmath}\,'}_{\vec{\jmath}} \delta^{\vec{\pi}'}_{\vec{\pi}} \; + \\ & - \frac{a^{d-4}}{2\; g^2\; \dim(U)} \sum_{\mathbf{y}} \sum_{r<s=1..d} \left(\langle \vec{\jmath}\,', \vec{\pi}' | U_{\mathbf{y},r,s} | \vec{\jmath}, \vec{\pi} \rangle + \langle \vec{\jmath}, \vec{\pi} | U_{\mathbf{y},r,s} | \vec{\jmath}\,', \vec{\pi}' \rangle \right) \end{aligned} \tag{8}$$

where the only non diagonal terms are just the afore computed expectation values of the plaquette operator.

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References

[1] G. Burgio, R. De Pietri, H. A. Morales-Técotl, L. F. Urrutia, and J. D. Vergara, *Nucl. Phys.* B, to appear. (hep-lat/9906036).

[2] G. Burgio, R. De Pietri, H. A. Morales-Técotl, L. F. Urrutia, and J. D. Vergara, *Nucl. Phys.* B *Proc. Suppl.* LATTICE99, to appear. (hep-lat/9909025).

[3] J. Kogut and L. Susskind, *Phys. Rev.* **11**, 395 (75).

[4] K.G.Wilson, *Phys. Rev.* **10**, 2445 (74).

[5] S. Mandelstam, *Phys. Rev.* D **19**, 2391 (79). B. Brügmann, *Phys. Rev.* D **43**, 566 (91). N.J. Watson, *Phys. Lett.* B **323**, 385 (94); *Nucl. Phys.* B *Proc. Suppl.* **39**, 224 (95) D. Schütte, Z. Weihong, and C.J. Hamer, *Phys. Rev.* D **55**, 2974 (97).

[6] A. P. Yutsis, I. B. Levinson, and V. V. Vanagas, *Mathematical Apparatus of the Theory of Angular Momentum*, Israel Program for Scientific Translations (Jerusalem, 1962). P. Cvitanoviè, *Group Theory*, Nordita Lecture Notes in Physics, (Copenhagen, 1994). N. Ja. Vilenkin and A. U. Klimyk. *Representation of Lie Groups and Special Functions, volume 1*, Kluwer Academic Publisher (Dordrecht, The Netherlands, 1993). D. M. Brink and G. R. Satchler, *Angular Momentum*, Claredon Press (Oxford, UK, 1968).

[7] J. J. De Swart. *Rev. of Modern Phys.* **35**, 916 (63). L. A. Shelepin, Z. Eksper. Teor. Fiz. **48**, 360 (65). F. J Archer, *Phys. Lett.* B **295**, 199 (92). N. Ja. Vilenkin and A. U. Klimyk, *Representation of Lie Groups and Special Functions, Recent Advances*, Kluwer Academic Publisher (Dordrecht, The Netherlands, 1995).